RESEARCH REPORT NO. 14, February 1964, NASA GRANT G246-62

CALCULATION OF LUNAR SURFACE AREA

by

David Friesen

Introduction

It is often useful to calculate the actual lunar surface area S represented by an area A on a photograph of the moon. To specialize the problem somewhat, we will restrict ourselves to an area A in the usual ξ, η projection plane, and assume that the boundaries of any area on a photograph are first expressed in terms of ξ, η.

A computer program has been written to facilitate the calculation for a class of areas A, and the use of it is described below.

Method of Calculation

The area A is projected onto the lunar surface, and the resulting area S is calculated by integrating the projection of A over the limits of A. In general, for any plane x,y and surface $z = f(x,y)$ the area S on z is related to the area A on x,y by

_ _ _ _ _ _ _ _ _ _ _ _ _

$$S = \iint_A \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \; dx \, dy \; . \qquad (1)*$$

In terms of our problem, we have

$$S = \iint_A \sqrt{1 + \left(\frac{\partial \zeta}{\partial \xi}\right)^2 + \left(\frac{\partial \zeta}{\partial \eta}\right)^2} \; d\xi \, d\eta \qquad (2)$$

We assume the moon to be spherical, and referring to figure 1, we calculate the partial derivitives:

$$\frac{\partial \zeta}{\partial \xi} = \pm \frac{\xi}{\sqrt{1 - \xi^2 - \eta^2}} \qquad (3)$$

$$\frac{\partial \zeta}{\partial \eta} = \pm \frac{\eta}{\sqrt{1 - \xi^2 - \eta^2}} \qquad (4)$$

where ζ is the third direction cosine of the set ξ, η, ζ.

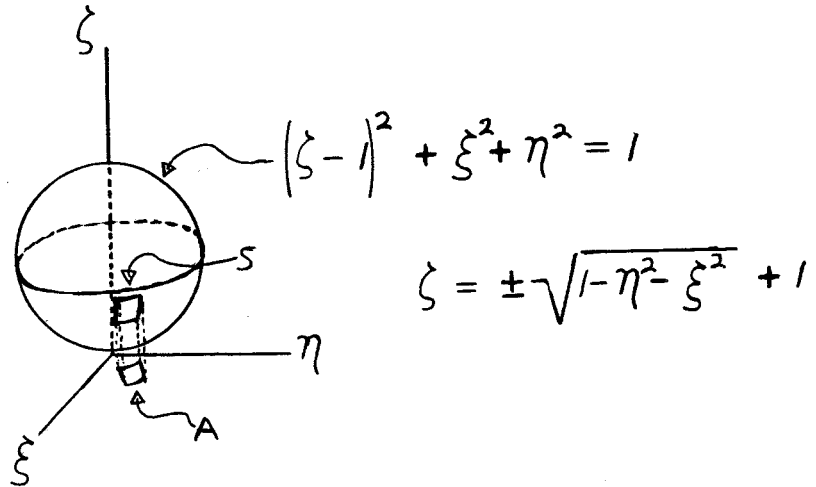*Thomas, <u>Calculus</u> & <u>Analytical</u> <u>Geometry</u> (2nd ed.), p. 552.

Figure 1.

Equation (2) then becomes:

$$S = \iint_A \sqrt{1 + \frac{\xi^2}{1-\xi^2-\eta^2} + \frac{\eta^2}{1-\xi^2-\eta^2}} \, d\xi \, d\eta \qquad \text{(5a)}$$

which can be simplified to

$$S = \iint_A \frac{1}{\sqrt{1-\xi^2-\eta^2}} \, d\xi \, d\eta \qquad \text{(5b)}$$

In (5b) the variables ξ, η are symmetric; we (arbitrarily) integrate with respect to ξ:

$$S = \int_{\eta_1}^{\eta_2} \left[\sin^{-1}\left(\frac{\xi}{\sqrt{1-\eta^2}}\right) \right]_{\xi=f_1(\eta)}^{\xi=f_2(\eta)} d\eta \qquad (6)**$$

Further integration of (6) depends upon the limits of ξ, and for almost any limits (including constant limits) it is either extremely difficult or impossible to perform the integration explicitly in terms of elementary functions.

When the limits $\xi_{1,2} = f_{1,2}(\eta)$ have been chosen, however, it is always possible to complete the second integration by numerical methods, such as Simpson's rule.

At this point we again stress the symmetry of ξ, η by writing (6) as

$$S = \int_{\alpha_1}^{\alpha_2} \left[\sin^{-1}\left(\frac{\beta}{\sqrt{1-\alpha^2}}\right) \right]_{\beta_1 = f_1(\alpha)}^{\beta_2 = f_2(\alpha)} d\alpha \qquad (7)$$

**C.R.C. <u>Standard</u> <u>Mathematical</u> <u>Tables</u> (12th ed.), integral No. 158 (p. 295).

where α, β can be either ξ, η or η, ξ (or a linear combination - see below), and can be freely interchanged in a systematic manner.

A computer program has been written which will calculate the area S on the lunar surface associated with an area A on the ξ, η plane, where A must be bounded by straight lines. The input to the computer consists of descriptions of areas bounded either by

a) 4 lines, 2 of which must be parallel,

or

b) 3 lines forming a triangle.

Rules for converting other areas to these forms are given in the following section.

Definition of Area on Plane

Areas on the ξ, η plane must be broken into areas of type (a) or (b) in figure 2.
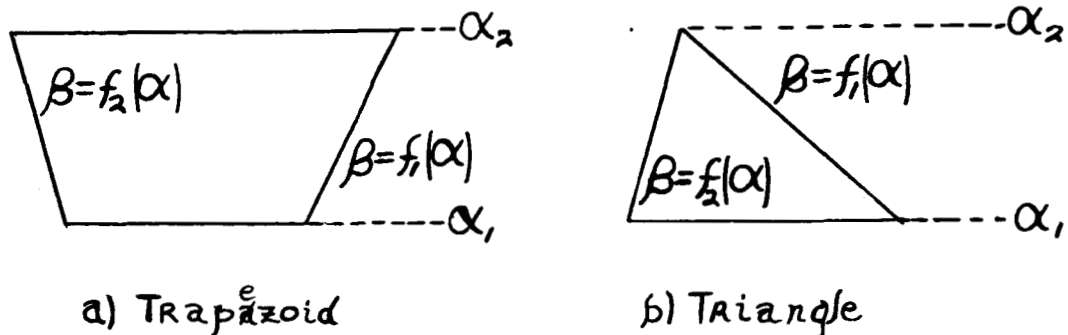

a) Trapezoid   b) Triangle

Figure 2 - Input Area types

Note that α_1, α_2 are lines of constant α . A typical area A

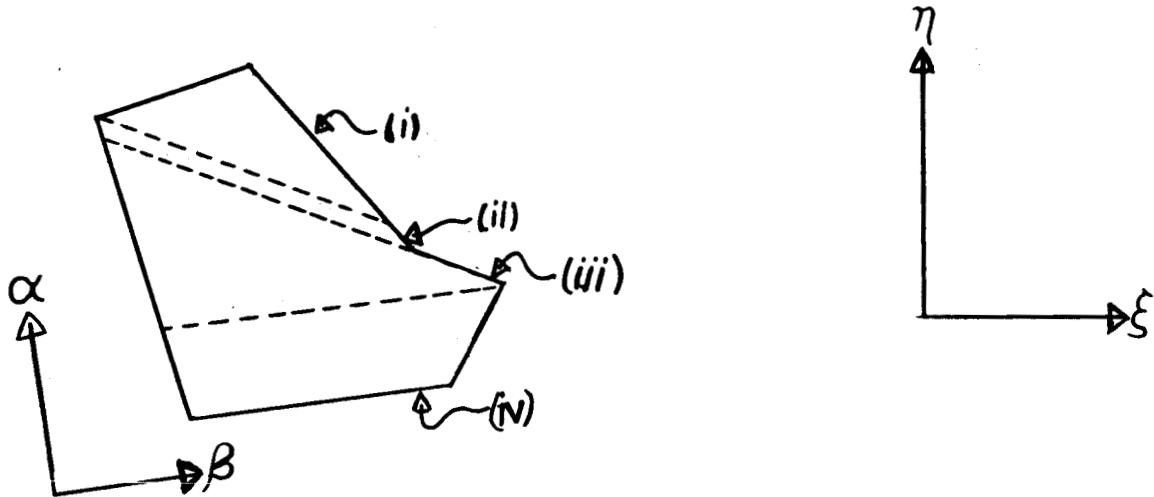is shown in figure 3, along with a possible division of it.

Figure 3 - Area to be computed

The boundaries of the sub-sections (i),...,(iv) are expressed as lines in terms of ξ, η . Then a set of α,β coordinates is chosen for <u>each</u> sub-section (i),...,(iv). The section boundaries (expressed in ξ, η) are then rotated by an orthonormal transformation so that either the parallel sides of the trapazoid or the base of the triangle is parallel to the α axis, (the boundaries now being expressed in α,β representation). The coordinate axes α,β are shown for sub-section (iv) in Figure 3.

The rotation is accomplished by either of the transformations

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{8a}$$

or

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{8b}$$

where θ is the angle between either ξ, α (in (8a)) or η, α (in (8b)).

The transformed boundary equations form the input to the computer, and in the example of figure 3 the lunar sur-face area S , of which A is the projection, is found by adding the areas obtained from sub-sections (i),...,(iv).

A simple example will clarify the work required.

Example 1:
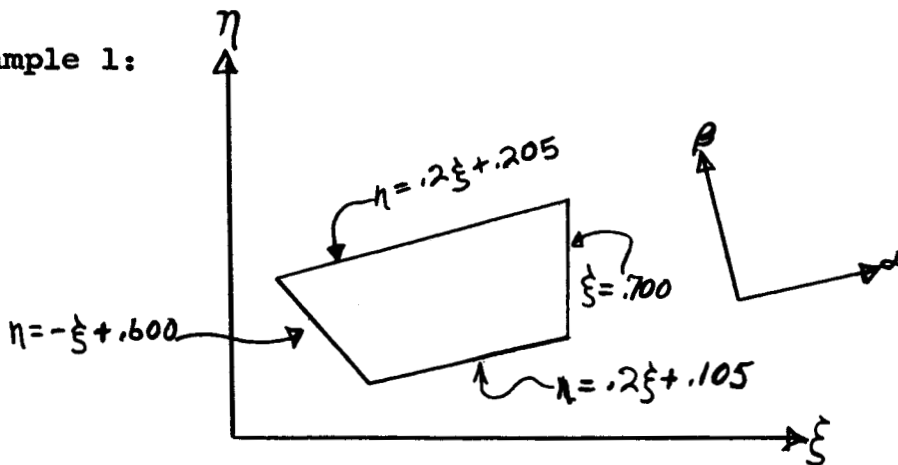
We will rotate ξ to α , employing (8a). α has been chosen parallel to $\eta = .2\,\xi\, + .105$

$\theta = \tan^{-1} .2 = .2$ radians

$\sin .2 = \quad .20$ Then using (8a)

$\cos .2 = \quad .98$

$\xi = \quad .98\alpha - .20\beta$

$\eta = \quad .20\alpha + .98\beta$

The transformed boundaries become:

$.20\alpha + .98\beta = .2(.98\alpha - .20\beta) + .105$

$1.02\beta = .105$

$\beta = .103$

and similarly $\beta = .201$

$\alpha = -.66\beta + .510$

$\alpha = .202\beta + .715$

On many occasions the α, β coordinate system can be chosen to coincide with the ξ, η system, and the work of making the transformations can thus be avoided.

Calculation of Area with a Computer

A computer program has been written which calculates
the areas S of sub-sections of an area A (as in figure 3).
The method is numerical integration of (7) using Simpson's
rule. In addition, the program can be instructed to total
sub-sections to give an S for an A such as in figure 3.
The program requires the following information:

    1)   limits of first integration of (7), in form

$$\beta_1 = A_1\alpha + B_1 \quad \text{and} \quad \beta_2 = A_2\alpha + B_2 \ ;$$

    2)   limits of second integration of (7), α_1 and α_2 ;

    3)   number of integration strips, N.

    The information is punched on cards in the following
format:

Columns: 1 - 10,	11 - 20,	21,	22 - 75
A_1	B_1	1	Anything
A_2	B_2	2	"
N	(blank)	3	"
α_1	α_2	0	Any identification of sec-tion being integrated

These are the control cards, and the number in column 21 is
the control code. There are four additional forms of input,
used to direct the program. When using the following cards,

columns 1 - 20 should be left blank.

Purpose	Col. 21	Col. 22 - 75
Accept input from typewriter	4	Anything
Total all areas computed from beginning of program or last total card.	5	Identification of totaled area
Repeat calculation of area with last α_1, α_2	0	Any identification

The final control card has control code 0 and any number

+1.0 in columns 1 - 10. It signals the end of the series

of control cards, and should be the last card in the data

deck. Input numbers are in FORTRAN F10.0 form; i.e., they

can be punched anywhere in columns 1 - 10 (or 11 - 20) with

a sign preceding them and a decimal point in the proper rel-

ative place. If no sign is punched, the number is assumed + .

Control codes are punched as a single digit in column 21, and

a blank in column 21 is considered a 0 . Columns 22 - 75

are ignored by the program, except when a control code 0 or 5

is encountered; the information is then printed (or punched)

along with the output as identification.

Values for the input data (A_1, α_1, etc.) remain as

set by first control cards until specifically reset by another

control card. Computation of an area occurs whenever a con-

trol code 0 is encountered, using the last-set values of

Note that control code 5 should be used to clear the total before summing a group of areas, unless the areas are the first ones to be calculated in a run. Repeat cards should not be used in a section to be totaled.

The program has been coded for both the IBM 1620 and the IBM 709-90-94 series. Coding in both cases is in FORTRAN II. Input is interchangeable between the two programs. Their peculiarities are discussed below.

IBM 1620 Program Operation

Normal operation consists of card input and punched output. Alternative of typewriter input (output) can be chosen by setting sense switches (see below) or using control code 4 in card input.

At least one each of control codes 1,2,3 (in any order) must precede the first control code 0. Only the first 18 columns of identification are used.

Large values of N should be avoided because of running time problems. A rough estimate for timing is:

$$\text{Running time (min.)} = \frac{N}{100}$$

Sense Switch Settings

SSW	On	Off
1	Input from typewriter	Input from cards
2	Print current value of iteration counter	Ignore
3	Type & punch output	Punch output
4	Correct typewriter input	Accept typewriter input

When using typewriter input, all necessary information for entry will be typed out.

IBM 709-90-94 Program Operations

Card input, printed output is the only mode of operation. Control code 4 is ignored when encountered. Large values of N can be used. Input cards can be in any (reasonable) order; if control code 0 is encountered before all other data has been read, the program will continue to read data and execute computation when enough data has been read. All 48 columns of identification are used.

A_1, α_1, N, etc. Thus a sequence of control cards:

A_1	B_1	1	
A_2	B_2	2	
N_1			
α_1	α_2	0	Area A (No. 26)
α_1'	α_2'	0	Area B (No. 27)
		5	Sum (A + B)
A_2'	B_2'	2	
α_1''	α_2''	0	Area C – Highland
N'		3	
		0	Repeat C
1.0		0	(End)

would cause output of

$S_1(A_1,B_1,A_2,B_2,N,\alpha_1,\alpha_2)$	AREA A (No. 26)
$S_2(" \ ," \ ," \ ," \ ,",\alpha',\alpha_2')$	AREA B (No. 27)
$S_1 + S_2$	SUM (A + B)
$S_3(A_1,B_1,A_2',B_2',N,\alpha_1'', \alpha_2'')$	AREA C – HIGHLAND
$S_4(" \ ," \ ," \ ," \ ,N',", \ " \ )$	REPEAT C

Summary of control cards

Operation	Data		Code	
	Col 1-10	11-20	21	22 – 75
Set Integration Limits	α_1	α_2	0	Identification
Calculation with last α_1, α_2			0	Identification
End	1.0		0	
Set integration limit	A_1	B_1	1	
Set integration limit	A_2	B_2	2	
Set integration grid	N		3	
Accept typewriter input			4	
Total			5	Identification

Further Reference

Flow charts, program listings, and test inputs are filed under "AREA PROG" in Lunar File, Astronomy Laboratory. Copies of the program can be obtained from:

Prof. Gerald S. Hawkins

Astronomy Department

Boston University

700 Commonwealth Avenue

Boston, Massachusetts 02215